Error Analysis of Semianalytic Displacement Derivatives for Shape and Sizing Variables

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For efficiency, the derivatives used in structural optimization are often calculated using a technique known as the semianalytic method. Here, we investigate the accuracy of this method for calculating static displacement derivatives with respect to both shape and sizing design variables. We show that the errors are entirely due to inaccuracies introduced during the finite-difference approximation of the stiffness matrix derivatives. Two types of errors are discussed. In the first case, the finite-difference error causes uniform scaling of the derivatives relative to their true values and the magnitude of the relative error depends only on the choice of the finite-difference parameter and the finite-difference formula. In the second case, errors in the finite-difference operation may lead to nonuniform errors in the displacement derivatives. The magnitudes of the relative errors depend not only on the finite-difference parameter and formula, but on the location of the element(s) associated with the design variable and the discretization of the structure. We demonstrate our error analysis using several example problems, including a representative automotive frame. We show for these problems that the relative errors can be adequately controlled through the choice of the finite-difference parameter.

Introduction

SENSITIVITY analysis is an integral part of the solution of the structural optimization problem. Consequently, much research work has focused on the efficient computation of static displacement derivatives and several formulations have been developed.[1] Although these methods differ in their approaches, the results of the computations are equivalent. As originally developed, these methods relied on analytical representations for the derivatives of the structural stiffness matrix K with respect to the design variables b_i. For some simple elements—beams and trusses, for instance—the formulation of K in terms of the b_i is well known, and it is relatively easy to derive analytic expressions for $\partial K/\partial b_i$; however, this is generally not possible for more complicated element formulations and for most shape design variables. In order to overcome these difficulties and facilitate the implementation of these methods into structural optimization programs[2] using general finite element analysis codes,[3,4] the semianalytic method[5,6] was developed. In this method, we approximate the matrix $\partial K/\partial b_i$ by finite difference. In so doing, the approach retains the simplicity and efficiency of the analytic derivative technique while generalizing it for use with standard finite element codes.

Although it is well known that the accuracy of the semianalytic method depends on the finite-difference parameter, a recent study[7] suggested that more fundamental inaccuracies exist in the method, particularly when used for shape optimization. Another study,[8] restricted to analyzing displacement derivatives of a cantilevered beam with respect to shape design variables, has argued qualitatively that these errors are related to nonuniform errors in the stiffness matrix sensitivities.

In this paper, we first develop a general framework for the error analysis. We show that the errors in the semianalytic method are merely the result of finite-difference errors in the pseudoload vector. Within this framework we develop analytical error expressions for derivatives with respect to both shape and sizing design variables for a cantilevered beam. Using these expressions, we show analytically that the errors in the semianalytic method are entirely accounted for by errors in the finite-difference approximation of $\partial K/\partial b_i$. We also show that the relative errors may occur nonuniformly throughout the structure for either type of design variable. Finally, we present computational results for a simple cantilevered beam and a three-dimensional automotive beam structure that confirm our analyses.

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Static Displacement Derivatives

Consider the discretized static equilibrium equations given by

$$Ku = p \tag{1}$$

where K is the system stiffness matrix, u is a vector of nodal displacements, and p is the vector of external nodal loads. Differentiating Eq. (1) with respect to the design variable b_i yields the following exact expression for the static displacement derivatives

$$\frac{\partial u}{\partial b_i} = -K^{-1}\frac{\partial K}{\partial b_i}u = -K^{-1}p_i^s \tag{2}$$

where, for convenience, it has been assumed that p is independent of b_i and where p_i^s is referred to as the pseudoload vector. Although the application of Eq. (2) is conceptually straightforward, its implementation in general finite-element codes is complicated by the need for an analytic representation for the derivative of the stiffness matrix with respect to b_i. This is especially true when b_i represents a structural shape design variable.

A more tractable method for calculating the displacement derivatives is obtained by replacing $\partial K/\partial b_i$ with a finite-difference approximation,[5,6] i.e.,

$$\frac{\partial K}{\partial b_i} \simeq \frac{\Delta K}{\Delta b_i} \tag{3}$$

Substituting Eq. (3) into Eq. (2) gives

$$\frac{\partial u}{\partial b_i} \simeq \frac{\Delta u}{\Delta b_i} = -K^{-1}\frac{\Delta K}{\Delta b_i}u = -K^{-1}\tilde{p}_i^s \tag{4}$$

where $\tilde{p}_i^s$ is the approximate pseudoload vector. This so-called semianalytic method [Eq. (4)] is clearly easier to implement than Eq. (2), requiring only some general scheme for caculating $\Delta K/\Delta b_i$. Unless otherwise stated, $\Delta K/\Delta b_i$ is obtained by forward finite difference, i.e.,

$$\frac{\Delta K}{\Delta b_i} = \frac{K(b + e_i \Delta b_i) - K(b)}{\Delta b_i} \tag{5}$$

where e_i is the ith unit vector and

$$\Delta b_i = cb_i \tag{6}$$

and $c > 0$ is the finite-difference parameter.

Accuracy of the Semianalytic Method

It is well known that the accuracy of the semianalytic method for computing static displacement derivatives is dependent on the choice of c in Eq. (6). A comparison of Eqs. (2) and (4) reveals that for $c \neq 0$ the errors in the displacement derivatives are a consequence of errors contained in the approximate pseudoload vector $\tilde{p}_i^s$. Theoretically, as c approaches zero, the accuracy of the semianalytic method approaches that of the analytic method. In practice, the choice of c is limited by the precision of the computer on which the method is implemented. As a result, for any reasonably small choice for the value of c, the displacement derivatives obtained by the semianalytic method will contain finite-difference error.

In order to characterize this error, first consider the case where, for some variable b_i, the structural stiffness matrix can be separated into two terms, as follows:

$$K = \bar{K} + f(b_i)K_{b_i} \tag{7}$$

where $\bar{K}$ and K_{bi} are independent of b_i, and $f(b_i)$ represents some function of b_i. For this case, the displacement derivatives are given by

$$\frac{\partial u}{\partial b_i} = -K^{-1}\frac{\partial f(b_i)}{\partial b_i}K_{b_i}u = -K^{-1}p_i^s \tag{8}$$

when calculated analytically, and by

$$\frac{\Delta u}{\Delta b_i} = -K^{-1}\frac{\Delta f(b_i)}{\Delta b_i}K_{b_i}u = -K^{-1}\tilde{p}_i^s \tag{9}$$

when the semianalytic method is used. Introducing a relative error measure

$$\epsilon_i = \left[\frac{\Delta f(b_i)}{\Delta b_i} - \frac{\partial f(b_i)}{\partial b_i}\right] \Big/ \frac{\partial f(b_i)}{\partial b_i} \tag{10}$$

to represent the accuracy of the finite-difference approximation of $\partial f(b_i)/\partial b_i$, one can write

$$\frac{\Delta f(b_i)}{\Delta b_i} = (1 + \epsilon_i)\frac{\partial f(b_i)}{\partial b_i} \tag{11}$$

Substituting Eq. (11) into Eq. (9) gives the following expression for the approximate pseudoload vector and displacement derivatives

$$\tilde{p}_i^s = (1 + \epsilon_i)p_i^s \tag{12}$$

$$\frac{\Delta u}{\Delta b_i} = (1 + \epsilon_i)\frac{\partial u}{\partial b_i} \tag{13}$$

It is clear from the preceding that when the conditions represented by Eq. (7) are satisfied, the semianalytic method yields displacement derivatives that are scaled with respect to the analytic derivatives. This is because $\tilde{p}_i^s$ is a simple scaling of p_i^s. As a result, the accuracy of the derivatives depends only on the accuracy of $\Delta f(b_i)/\Delta b_i$.

If, as is often the case, the conditions represented by Eq. (7) are not satisfied then the errors in the displacement derivatives may have a significantly different form. Consider the following expression for the finite-difference approximation to $\partial K/\partial b_i$

$$\frac{\Delta K}{\Delta b_i} = \frac{\partial K}{\partial b_i} + E_i \tag{14}$$

where E_i denotes a matrix of errors resulting from the finite-difference operations on K. Substituting Eq. (14) into Eq. (4) and simplifying gives the following approximation for the displacement derivatives:

$$\frac{\Delta u}{\Delta b_i} = -K^{-1}\tilde{p}_i^s = \frac{\partial u}{\partial b_i} - K^{-1}E_i u \tag{15}$$

where

$$\tilde{p}_i^s = p_i^s + E_i u \tag{16}$$

In this case, $\tilde{p}_i^s$ is not necessarily a simple scaling of p_i^s. Geometrically, this means that the approximate pseudoload vector may have the wrong shape as well as the wrong magnitude. Also, since in this case $\tilde{p}_i^s$ is a function of both E_i and u, the accuracy of the displacement derivatives depends not only on the errors in the entries of $\Delta K/\Delta b_i$ but also on their interactions with u. Therefore, the accuracy of the derivatives may depend on the number of elements in the structural model and the location, within the model, of the element(s) dependent on b_i.

Error Analysis for a Cantilevered Beam

To illustrate the concepts set forth in the preceding section, consider the initially uniform cantilevered beam of length L, shown in Fig. 1. The beam has a rectangular cross section and is subject to force and moment loads at the tip. It is modeled with N beam type finite elements, each having a length $\ell = L/N$. In the following, we investigate the accuracy of the tip displacement derivatives with respect to sizing and shape design variables ($\Delta u'/\Delta b_i$) when calculated via the semianalytic method.

Sizing Design Variables—Planar Case

For the cantilevered beam just described, we consider the derivative of the tip displacement with respect to the height h_n of the nth element. Initially, we take $F_y = 0$, $M_y = 0$ so that all displacements occur in the x-z plane. The element level stiffness matrix for the x-z plane is given by

$$K_n^e = \frac{2EI_n}{\ell_n^3}\begin{bmatrix} 6 & -3\ell_n & -6 & -3\ell_n \\ & 2\ell_n^2 & 3\ell_n & \ell_n^2 \\ & & 6 & 3\ell_n \\ & \text{sym} & & 2\ell_n^2 \end{bmatrix} \tag{17}$$

where

$$I_n = (w_n h_n^3)/12 \tag{18}$$

Clearly, in this case, K can be written in the form of Eq. (7), where

$$f(h_n) = h_n^3 \tag{19}$$

Consequently, the relative error in the tip displacement derivative (ϵ^n) given by Eq. (10) is

$$\epsilon^{h_n} = (c^2 + 3c)/3 = \epsilon^h \tag{20}$$

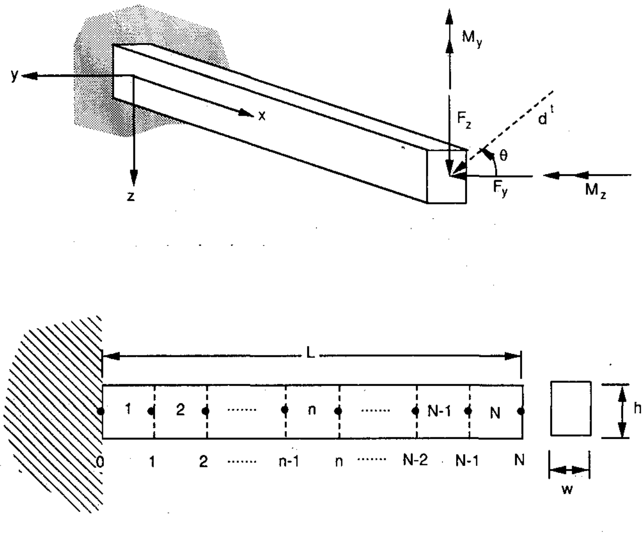

Fig. 1 Cantilevered beam finite element model.

where c is the forward finite-difference parameter. Note that the relative error is quite small (0.1%) for values of c less than 0.001. It should also be noted that the error is independent of both the number of elements used to model the beam and the location of the element along the beam.

Shape Design Variables

In this case, the accuracy of the tip displacement derivatives with respect to the length ℓ_n of the nth element is investigated. Since the element level stiffness matrix [Eq. (17)] contains different powers of ℓ_n, it will not be possible to write K in the form of Eq. (7). Therefore, using Eq. (4), we will derive an exact expression for the semianalytic tip displacement derivative $\Delta u^t/\Delta\ell_n$ in order to assess its accuracy. In this case, working from Eq. (4), $\Delta u^t/\Delta\ell_n$ can be obtained from

$$\frac{\Delta u^t}{\Delta \ell_n} = -r_n^T \frac{\Delta K_n^e}{\Delta \ell_n} u_n \tag{21}$$

where u_n is a vector of displacements associated with the nth element and $\Delta K_n^e/\Delta\ell_n$ is a forward difference approximation to $\partial K_n^e/\partial\ell_n$. Letting r^T be the row of K^{-1} corresponding to the tip displacement degree of freedom, r_n^T is a row vector containing only the entries of r^T, associated with the degrees of freedom of the nth element.

To evaluate Eq. (21), first consider the following expressions for the displacements and rotations along the length of a uniform cantilevered beam subject to tip loads $F_y = 0$, $M_y = 0$, $F_z = F$, and $M_z = M$:

$$u(x) = \frac{(FL - M)x^2}{2EI} - \frac{Fx^3}{6EI} \tag{22}$$

$$\theta(x) = \frac{Fx^2}{2EI} + \frac{(M - FL)x}{EI} \tag{23}$$

Now, let the beam be composed of N elements of length $\ell_n = \ell = L/N$ numbered from 1 to N, starting at the root (Fig. 1). If the nodes are numbered from 0 to N, starting at the root, then the nth node has location

$$x = (nL)/N = n\ell \tag{24}$$

Substituting Eq. (24) into Eqs. (22) and (23) yields the following expressions for u and θ at node n:

$$u(n) = \left[\frac{(FL - M)}{2EI} - \frac{Fn\ell}{6EI}\right] n^2\ell^2; \qquad n = 0,\ldots,N \tag{25}$$

$$\theta(n) = \left[\frac{Fn\ell}{2EI} + \frac{(M - FL)}{EI}\right] n\ell; \qquad n = 0,\ldots,N \tag{26}$$

These equations represent the solution to Eq. (1) for any given discretization of the beam. By recognizing that

$$u = K^{-1} p \tag{27}$$

implies

$$K_{ij}^{-1} = K_{ji}^{-1} = \frac{\mathrm{d}u_j}{\mathrm{d}p_i} \tag{28}$$

it is possible to obtain any entry of K^{-1}. Therefore, by symmetry of K^{-1} we may compute

$$r_n^T = \frac{\mathrm{d}}{\mathrm{d}F}\begin{Bmatrix} u(n-1) \\ \theta(n-1) \\ u(n) \\ \theta(n) \end{Bmatrix} = \begin{Bmatrix} \dfrac{(n-1)^2\ell^2}{6EI}[3L - (n-1)\ell] \\ \dfrac{(n-1)\ell}{2EI}[(n-1)\ell - 2L] \\ \dfrac{n^2\ell^2}{6EI}(3L - n\ell) \\ \dfrac{n\ell}{2EI}(n\ell - 2L) \end{Bmatrix} \tag{29}$$

Finally, given that

$$\frac{\Delta K_n^e}{\Delta \ell_n} = \frac{2EI}{\ell_n^2}\begin{bmatrix} -\dfrac{18}{\ell_n^2}C_1 & \dfrac{6}{\ell_n}C_2 & \dfrac{18}{\ell_n^2}C_1 & \dfrac{6}{\ell_n}C_2 \\ & -2C_3 & -\dfrac{6}{\ell_n}C_2 & -C_3 \\ & & -\dfrac{18}{\ell_n^2}C_1 & -\dfrac{6}{\ell_n}C_2 \\ \text{sym} & & & -2C_3 \end{bmatrix} \tag{30}$$

where

$$C_1 = \frac{(c^2/3) + c + 1}{(c+1)^3} \tag{31}$$

$$C_2 = \frac{(c/2) + 1}{(c+1)^2} \tag{32}$$

$$C_3 = 1/(1+c) \tag{33}$$

and that u_n can be obtained from Eqs. (25) and (26), where

$$u_n = \begin{Bmatrix} u(n-1) \\ \theta(n-1) \\ u(n) \\ \theta(n) \end{Bmatrix} \tag{34}$$

Equation (21) can be evaluated for $\Delta u^t/\Delta\ell_n$. To simplify the final result, consider the case where $F_z = F = 0$. Equations (25) and (26) become

$$u(n) = -(Mn^2\ell^2)/(2EI) \tag{35}$$

$$\theta(n) = (Mn\ell)/(EI) \tag{36}$$

Using Eqs. (29–36), Eq. (21) gives the following expression for the tip displacement derivative:

$$\frac{\Delta u^t}{\partial \ell_n} = -\frac{ML}{EI}[2N - c^2(2n - 2N - 1) + c(12n^3 - 24n^2N - 18n^2 + 24nN + 8n - 2N - 1)]/[2N(c+1)^3]$$

$$= -\frac{ML}{EI}[\epsilon(c,n,N) + 1] \tag{37}$$

where $\epsilon(c,n,N) = \epsilon_n$ is the relative error in $\Delta u^t/\Delta\ell_n$. Clearly, $\Delta u^t/\Delta\ell_n$ depends not only on the accuracy of $\Delta K_n^e/\Delta\ell_n$, as determined by c, but also on the discretization of the beam and on the element number n.

In Refs. 7 and 8, the semianalytic derivatives $\Delta u^t/\Delta L$ are discussed. This quantity is based on perturbations of all elements in the beam, such that the quantity ΔL is distributed evenly among all elements. Noting that

$$\frac{\partial u^t}{\partial L} = \sum_{n=1}^{N} \frac{\partial u^t}{\partial \ell_n}\frac{\partial \ell_n}{\partial L} = \frac{1}{N}\sum_{n=1}^{N}\frac{\partial u^t}{\partial \ell_n} \tag{38}$$

we can show that the relative error in $\Delta u^t/\Delta L$ is equivalent to the average error in $\Delta u^t/\Delta\ell_n$, which is given by

$$\epsilon_{\text{avg}} = \frac{1}{N}\sum_{n=1}^{N}\epsilon_n = \frac{1}{N}\sum_{n=1}^{N}\frac{\dfrac{\Delta u^t}{\Delta \ell_n} - \dfrac{\partial u^t}{\partial \ell}}{\partial u^t/\partial \ell_n} \tag{39}$$

$$= \frac{1-(c/2)(5N^2-4-2/N-c)}{(c+1)^3} - 1 \tag{40}$$

Equation (40) shows that for small values of c, ϵ_{avg} is approximately proportional to cN^2.

For comparison with Ref. 7, Fig. 2 shows a plot of the average relative errors predicted, using Eq. (40) for various choices of N and c. As the number of elements increases, the average error also increases for all values of c. Note that Fig. 2 is in complete agreement with the numerical results of Ref. 7. Thus, their errors are completely explained by the inaccuracies in the pseudoload vectors resulting from the finite-difference approximation of $\partial K/\partial\ell$, and not, as suggested in Ref. 7, due to the inability of an Euler beam to properly represent derivatives with respect to length. For $c = 0.01$, errors of nearly 1000% occur as the number of elements increases to 20. However, for $c = 0.00001$, the average error is less than 1.0%, indicating that, in this case, the error can be controlled simply by proper choice of c.

As discussed in Refs. 7 and 8, the central difference formula (9) may be an effective way to control these errors. Replacing the forward difference error factors [Eqs. (31–33)] with the corresponding factors for central differences, we obtain the central difference analog to Eq. (37):

$$\frac{\Delta u^t}{\Delta \ell_n} = -\frac{ML}{EI}\left[\frac{N - Ac^4 - Bc^2}{N(1+c)^3(1-c)^3}\right] \tag{41a}$$

where

$$A = 6n^3 - 12n^2N - 9n^2 + 12nN + 7n - 4N - 2 \tag{41b}$$

$$B = 18n^3 - 36n^2N - 27n^2 + 36nN + 13n - 7N - 2 \tag{41c}$$

In Fig. 3, we have plotted the average errors obtained from Eqs. (41) as a function of N and c (solid curves) along with the errors obtained from Eq. (37) (dashed curves). The central difference formulation significantly reduced the error. In fact, for $c = 0.001$ and $N = 20$, the average error using central differences is only 0.3%, whereas the worst element has a relative error of only 0.7%. For $N = 20$ and $c = 0.00001$, the average error is reduced to 0.00003% and the relative error in the worst element is 0.00007%. This clearly indicates the potential of the central difference formula for reducing the errors.

In the next section we will discuss our analytic results in detail, comparing the relative errors in the semianalytic tip derivatives predicted from Eq. (37) with those obtained by computation from a structural optimization code.

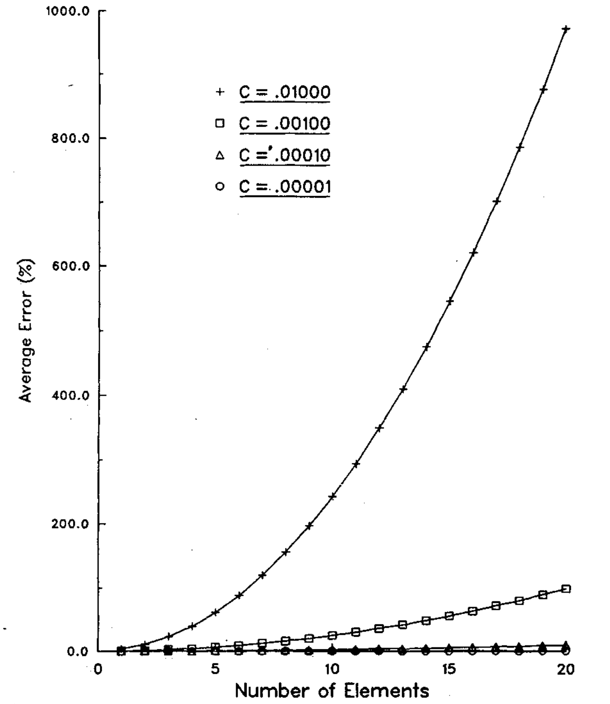

Fig. 2 Predicted average error of tip displacement derivative with respect to element length.

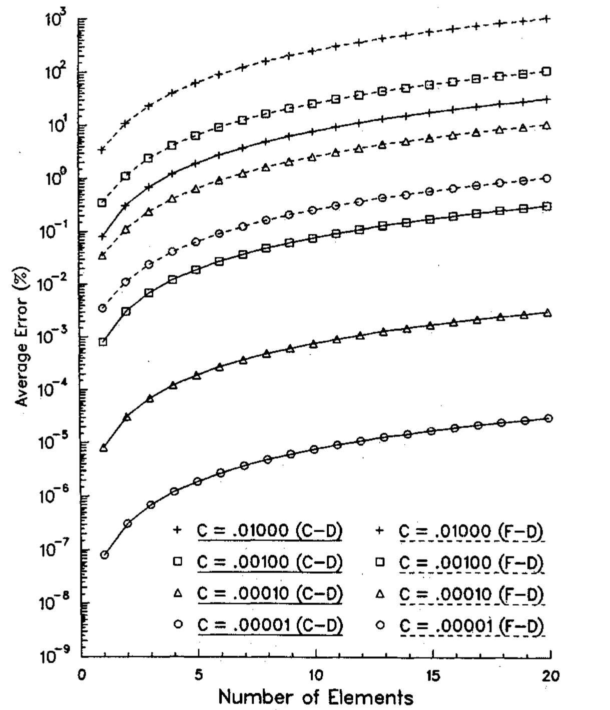

Fig. 3 Average error of tip displacement derivative with respect to element length: central difference vs forward difference formulas.

Sizing Design Variables—Nonplanar Case

Next, we consider the initially uniform cantilevered beam with multiple tip loadings in order to demonstrate that nonuniform errors may occur even for sizing variables. For simplicity, we again analyze the beam shown in Fig. 1, which has principal axes aligned with the global system and loadings applied in the principal planes. We will consider the derivative of the tip displacement d^t, which occurs at angle θ with the y axis. If d^t has components u^t and v^t, we may write

$$d^t = u^t\cos\theta + v^t\sin\theta \tag{42}$$

and it follows that the exact and semianalytic derivatives with respect to variable b_i may be written

$$\frac{\partial d^t}{\partial b_i} = \frac{\partial u^t}{\partial b_i}\cos\theta + \frac{\partial v^t}{\partial b_i}\sin\theta \tag{43}$$

and

$$\frac{\Delta d^t}{\Delta b_i} = \frac{\Delta u^t}{\Delta b_i}\cos\theta + \frac{\Delta v^t}{\Delta b_i}\sin\theta \tag{44}$$

For sizing variables we will show that the y and z components of the semianalytic derivatives can be written in the form of Eq. (13) as

$$\frac{\Delta u^t}{\Delta b_i} = (1+\varepsilon_i^u)\frac{\partial u^t}{\partial b_i} \tag{45}$$

$$\frac{\Delta v^t}{\Delta b_i} = (1+\varepsilon_i^v)\frac{\partial v^t}{\partial b_i} \tag{46}$$

It follows that the relative error is given by

$$\varepsilon_i^r = \frac{\dfrac{\Delta d^t}{\Delta b_i} - \dfrac{\partial d^t}{\partial b_i}}{\dfrac{\partial d^t}{\partial b_i}} = \frac{\varepsilon_i^u \dfrac{\partial u^t}{\partial b_i}\cos\theta + \varepsilon_i^v \dfrac{\partial v^t}{\partial b_i}\sin\theta}{\dfrac{\partial u^t}{\partial b_i}\cos\theta + \dfrac{\partial v^t}{\partial b_i}\sin\theta} \tag{47}$$

For sizing variables, the b_i will generally be associated with a single element, say n, and, thus, the relative error may vary from element to element since ϵ_i^u, ϵ_i^v, $\partial u^t/\partial b_i$, and $\partial v^t/\partial b_i$ may vary.

Uniform Error

We note that several simplifications to the general nonplanar case may lead to uniform relative error, even though the separability test [Eq. (7)] for the stiffness matrix is not satisfied. If $\epsilon_i^u = \epsilon_i^v = \epsilon$, a constant, Eq. (47) shows that no variation occurs with element number. Also, if the beam is uniform, $\epsilon_i^u = \epsilon^u$, $\epsilon_i^v = \epsilon^v$, and $\theta = i\pi/2$ for $i = 0,1,2...$, we will also have uniform ϵ^r. This can be seen by simplifying Eq. (47) for either $\sin\theta = 0$ or $\cos\theta = 0$. Here, although the global stiffness matrix is not separable, the problem degenerates to a planar problem in either u or v, which can be handled as described earlier.

Uniformity of relative error may also occur due to loading. To see this, we must consider the forms of $\Delta u^t/\Delta b_i$ and $\Delta v^t/\Delta b_i$. We begin by verifying Eqs. (45) and (46) for the cantilevered beam with the sizing variables shown in Fig. 1. Since the matrix equations for the y and z directions differ only in the moments of inertia, we can follow the preceding analysis for sizing variables. Assuming variable b_i occurs in element n, we may write Eq. (17) as

$$K_n^e = I_n \bar{K}_n^e \tag{48}$$

where $\bar{K}_n^e$ is not a function of b_i. It is clear that

$$K_n^y = I_n^y \bar{K}_n^e \tag{49}$$

$$K_n^z = I_n^z \bar{K}_n^e \tag{50}$$

where K_n^y and K_n^z are the element level stiffness matrices for the y and z directions, respectively, and $I_n^y = w_n h_n^3/12$, $I_n^z = h_n w_n^3/12$. Clearly, since $\bar{K}_n^e$ is independent of h_n, Eqs. (45) and (46) hold.

To determine the semianalytic derivatives $\Delta u^t/\Delta b_i$ and $\Delta v^t/\Delta b_i$, we also need expressions for $\partial u^t/\partial b_i$ and $\partial v^t/\partial b_i$. From Eqs. (2), (49), (50), and the definition of r_n, it follows that

$$\frac{\partial u^t}{\partial h_n} = -r_n^T \frac{\partial K_n^y}{\partial h_n} u_n \tag{51}$$

$$\frac{\partial v^t}{\partial h_n} = -r_n^T \frac{\partial K_n^z}{\partial h_n} v_n = -\alpha_n r_n^T \frac{\partial K_n^y}{\partial h_n} v_n \tag{52}$$

where $\alpha_n = 3h_n^2/w_n^2$. This may be further simplified if the y and z displacement fields are related by a simple scaling of the form $v(x) = \beta u(x)$, with β a constant. This will occur when the beam is uniform and the loads in the y and z directions are related by

$$P_z = \begin{Bmatrix} F_z \\ M_z \end{Bmatrix} = \gamma \begin{Bmatrix} F_y \\ M_y \end{Bmatrix} = \gamma P_y \tag{53}$$

with γ a scalar. In this case,

$$\frac{\partial v^t}{\partial h_n} = -\alpha_n r_n^T \frac{\partial K_n^y}{\partial h_n} v_n = -\alpha_n \beta r_n^T \frac{\partial K_n^y}{\partial h_n} u_n = \delta_n \frac{\partial u^t}{\partial h_n} \tag{54}$$

so that Eq. (47) simplifies to

$$\epsilon_n^r = \frac{\epsilon_n^u \cos\theta + \epsilon_n^v \delta_n \sin\theta}{\cos\theta + \delta_n \sin\theta} \tag{55}$$

If the beam is uniform $\epsilon_n^u = \epsilon^u$, $\epsilon_n^v = \epsilon^v$, and $\delta_n = \delta$, so that the relative error is simply

$$\epsilon^r = \frac{\epsilon^u \cos\theta + \epsilon^v \delta \sin\theta}{\cos\theta + \delta \sin\theta} \tag{56}$$

and is not a function of n. Again, note that, although the error is uniform, this problem does not meet the stiffness matrix separability test for h_n, which was illustrated in the general derivation.

Nonuniform Error

If none of the preceding simplifications holds, nonuniform errors will generally occur. Specifically, for the cantilevered beam, if $P_z \neq \gamma P_y$ and displacements are nonplanar, ϵ_n^r retains the general form of Eq. (47) and will vary depending on the element. This occurs because

$$\frac{\partial u(x)}{\partial h_n} \neq \delta_n \frac{\partial v(x)}{\partial h_n} \tag{57}$$

As an example, consider loading a uniform beam of length L composed of N elements of length $\ell = N/L$, with both a tip moment M_z, and a tip force F_y. The exact displacement fields are given by

$$u(x) = \frac{F_y}{EI_z}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \tag{58}$$

$$v(x) = -\frac{M_z}{2EI_y}x^2 \tag{59}$$

Using these expressions, we may determine $\partial u^t/\partial h_n$ and $\partial v^t/\partial h_n$ — the exact derivatives of the tip displacements with respect to changes in the height of element n.

$$\frac{\partial v^t}{\partial h_n} = \frac{18 M_z L^2}{E w_n h_n^4 N^2}[1 + 2(N-n)] \tag{60}$$

$$\frac{\partial u^t}{\partial h_n} = -\frac{4F_y L^3}{E h_n^2 w_n^3 N^3}[3(N-n)^2 + 3(N-n) + 1] \tag{61}$$

From these expressions, we can derive the semianalytic derivatives

$$\frac{\Delta v^t}{\Delta h_n} = \epsilon_n^v \frac{\partial v^t}{\partial h_n} \tag{62}$$

and

$$\frac{\Delta u^t}{\Delta h_n} = \epsilon_n^u \frac{\partial u^t}{\partial h_n} \tag{63}$$

Clearly, since

$$\frac{\partial v^t}{\partial h_n} = f\left(\frac{1}{h_n^4}, n\right) \tag{64}$$

$$\frac{\partial u^t}{\partial h_n} = f\left(\frac{1}{h_n^2}, n\right) \tag{65}$$

it follows that $\epsilon_n^u \neq \epsilon_n^v$ for any finite-difference parameter $c \neq 0$. From Eqs. (60) and (61), it is clear that the dependence on the number of elements N and element number n does not factor out of Eq. (47) and, thus, ϵ_n^r is a function of both N and n.

Numerical Results

In order to demonstrate numerically the analytical results just presented, consider two example problems. The first is a cantilevered beam. For this problem, numerical results are generated and compared with the analytical results developed in the preceding section. The second example, a typical automobile frame structure, demonstrates the qualitative results of the general error analysis for a more practical problem. Numerical results were generated using the structural optimization code described in Ref. 10.

Cantilevered Beam Problem

For the cantilevered beam problem shown in Fig. 1, we will assess the accuracy of the semianalytic method when used to calculate the tip displacement derivatives with respect to the height and length of the elements making up the beam. First, consider a beam composed of 20 elements and take $L = 1000$ cm, $M_z = 435.2$ Nt-m, $F_z = 0.0$ Nt, $M_y = 0.0$ Nt-m, and $F_y = 0.0$ Nt. Each element has length $\ell_n = 50.0$ cm, height $h_n = 8.0$ cm, and width $w_n = 5.0$ cm. For this case, the derivatives of the tip displacement with respect to the element height was calculated for each of the 20 elements, for four different values of the finite-difference parameter c. The derivatives were then normalized by their true values and the results plotted in Fig. 4. Note that the error in the derivatives is uniform and depends only on c. Also, the error is exactly equal to that predicted by Eq. (20).

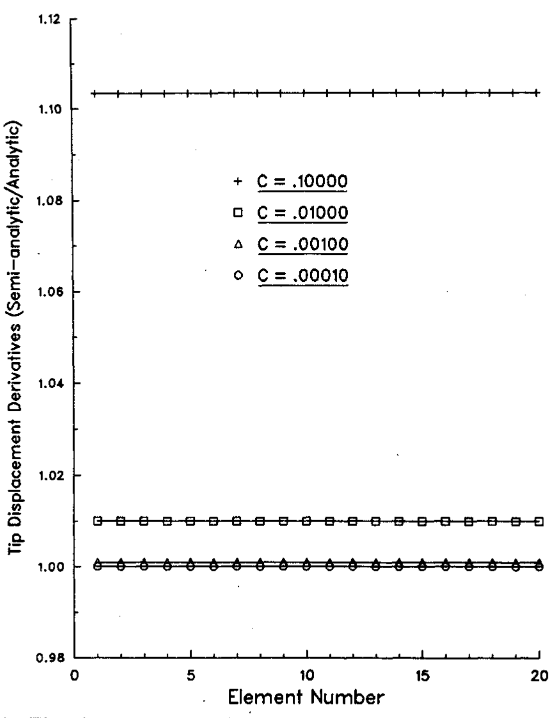

Fig. 4 Tip displacement derivatives with respect to element height (computed, $N = 20$).

For the second case, we again consider the beam just described with the following discretizations: 5 elements ($\ell_n = 200.0$ cm), 10 elements ($\ell_n = 100.0$ cm), 15 elements ($\ell_n = 66.667$ cm), 20 elements ($\ell_n = 50.0$ cm). Here, the tip displacement derivative with respect to ℓ_n is calculated for each element, for each of the four different beam discretizations. In Fig. 5, both the computed and predicted tip displacement derivatives are shown (normalized by the true derivatives) for all of the elements and for each discretization. The finite-difference parameter is $c = 0.01$. The predicted values of the derivatives were generated from Eq. (37). As expected, the relative error depends on both the number of elements (N) used to model the beam and on the location of the element within the beam (n). For this problem, the error increases with N, and increases as we move along the beam from the root to the tip. Note, also, that the computed errors are in complete agreement with the predicted errors. Figures 6 and 7 show the computed and predicted normalized derivatives for $c = 0.001$ and $c = 0.00001$. Again, the results demonstrate that the accuracy of the derivatives depends on both n and N. In Fig. 8, we show the results corresponding to $c = 0.01$, 0.001, 0.0001, and 0.00001, when the beam contains

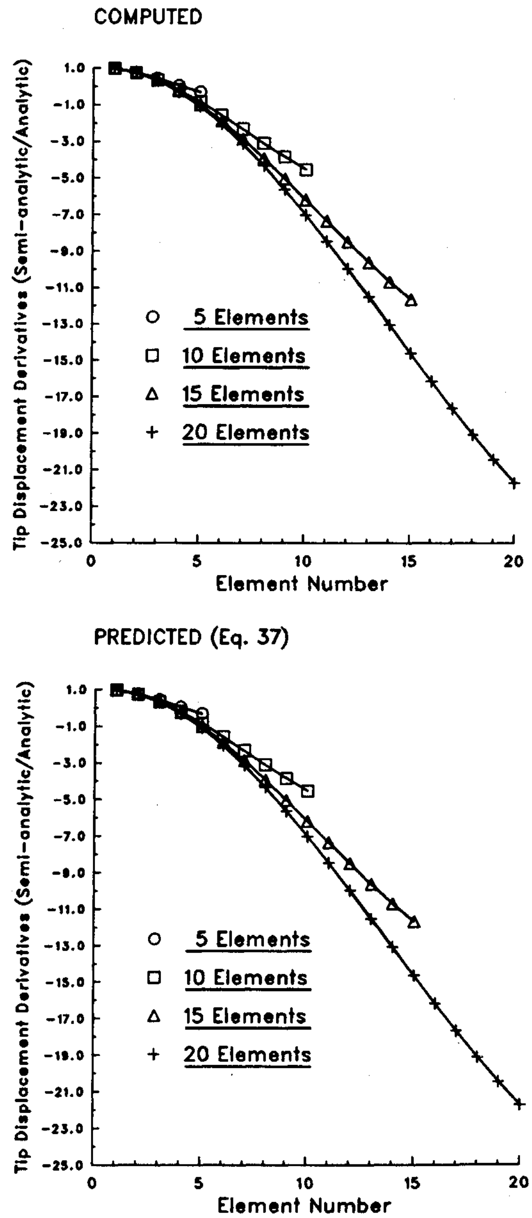

Fig. 5 Comparison of computed and predicted values of tip displacement derivatives with respect to element length ($c = 0.01$).

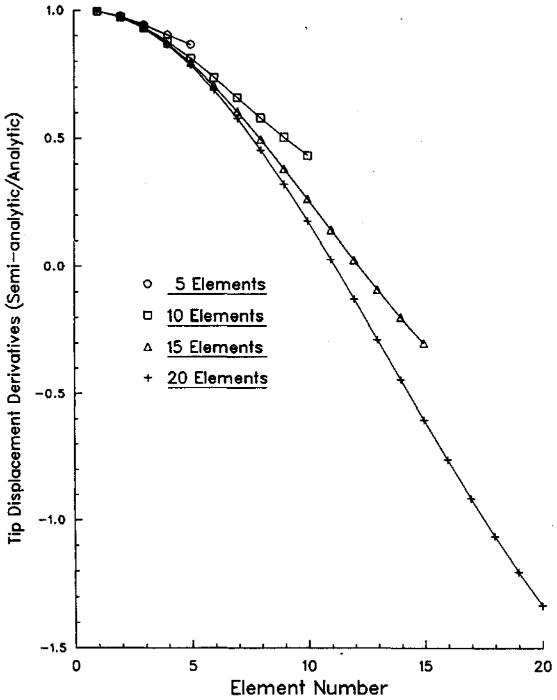

Fig. 6 Tip displacement derivatives with respect to element length ($c = 0.001$).

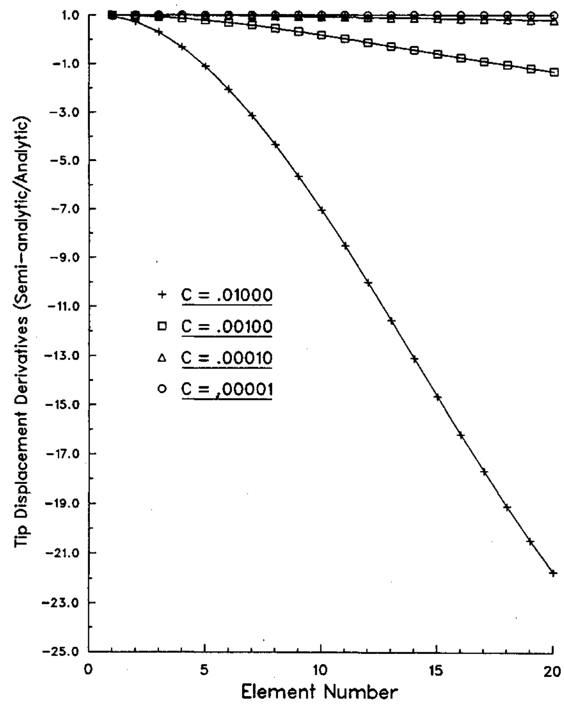

Fig. 8 Tip displacement derivatives with respect to element length ($N = 20$).

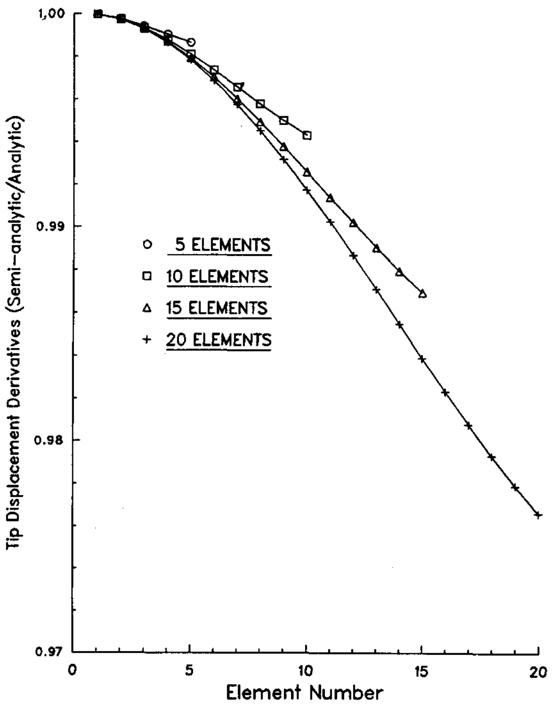

Fig. 7 Tip displacement derivatives with respect to element length ($c = 0.00001$).

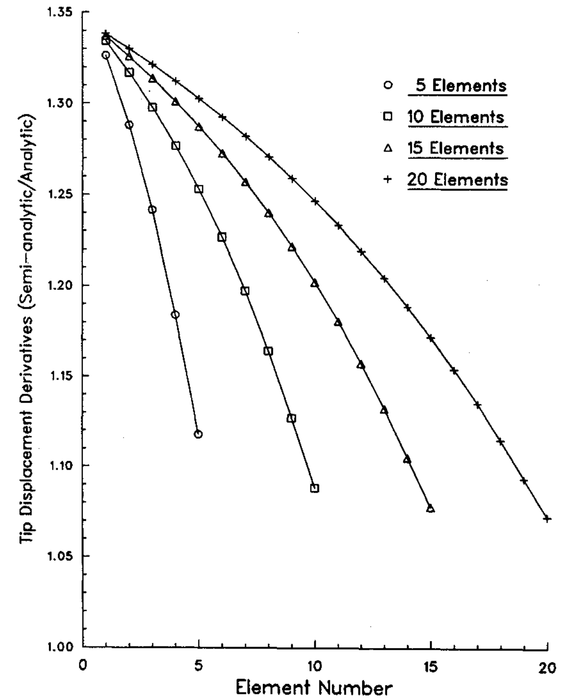

Fig. 9 Tip displacement derivatives with respect to element height–nonuniform error ($c = 0.1$).

20 elements. The magnitude of the relative error decreases rapidly with decreasing values of c. Clearly, for values of c greater than 0.0001, the accuracy of the derivatives is not acceptable. In fact, when $c = 0.01$, the sign of the derivatives is incorrect in most cases. However, acceptable error (less than 3.0%) is obtained for $c = 0.00001$.

Finally, consider the cantilevered beam shown in Fig. 1 and loaded with $F_y = -307.73$ Nt, $M_z = -307.73$ Nt-m, $F_z = 307.73$ Nt, $M_y = -307.73$ Nt-m. This loading condition produces displacement fields in the y and z directions, which are not simple scalings of each other. Consequently, from our preceding analysis we expect that the errors in the derivatives of the tip displacement d' (for $\theta = \pi/4$), with respect to the element heights, will depend on n and N. The numerical results for this problem are shown in Figs. 9–11. Four different beam discretizations ($N = 5$, 10, 15, and 20) and three values of the finite-difference parameter ($c = 0.1$, 0.01, and 0.001) are considered. In this case, the derivatives are normalized by values calculated using the analytical formulation of Eq. (2). Note that in this case the errors in the derivatives depend, as expected, on c, n, and N. The magnitudes of the errors are, however, significantly less for a given value of c than we found for the derivatives with respect to element lengths. Figure 11 shows that for $N = 20$, acceptable accuracy is obtained for values of c as large as 0.01.

Automobile Frame Problem

To study the accuracy of the semianalytic method for more practical problems, consider the half model of a typical automobile frame,[2] shown in Fig. 12. The structural model consists of 33 three-dimensional beam-type finite elements, each having a rectangular cross section. The structure is simply supported at the front suspension attachment point A and loaded in the vertical direction at the rear suspension attachment point B. Boundary conditions are applied to the center line grid points to force an antisymmetric structural response. The net effect of the loading and boundary conditions is to cause torsion of the structure about the centerline. In this case, we calculate the derivatives of the vertical displacement at point C with respect to the thickness, width, height, and length of each element in the structure and compare these against those calculated numerically, using Eq. (2).

Figure 13 shows the percent error in the semianalytic derivatives for the element thicknesses, widths, heights, and lengths. For each case, three different values of c are considered, and the maximum, minimum, and average errors are plotted. In all cases, the accuracy of the derivatives varies from element to element.[11] For the thickness derivative, the largest error (10%) occurs for element number 24 when $c = 0.1$. Acceptable accuracy for all elements is obtained for values of c less than 0.01. Similar results are shown for the derivatives with respect to the element widths and heights. However, the magnitudes of the errors are somewhat greater.

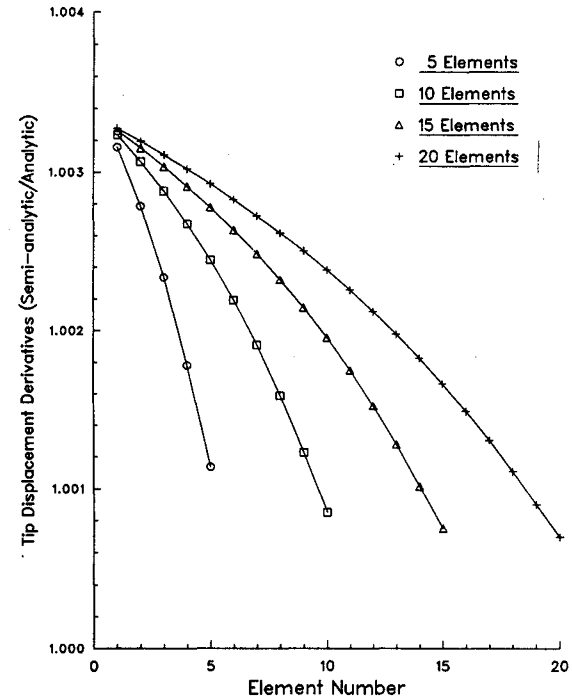

Fig. 10 Tip displacement derivatives with respect to element height – nonuniform error ($c = 0.001$).

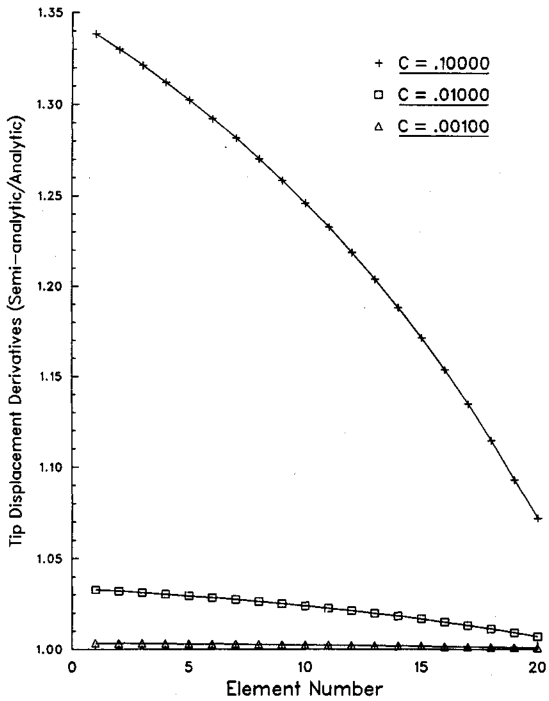

Fig. 11 Tip displacement derivatives with respect to element height – nonuniform error ($N = 20$).

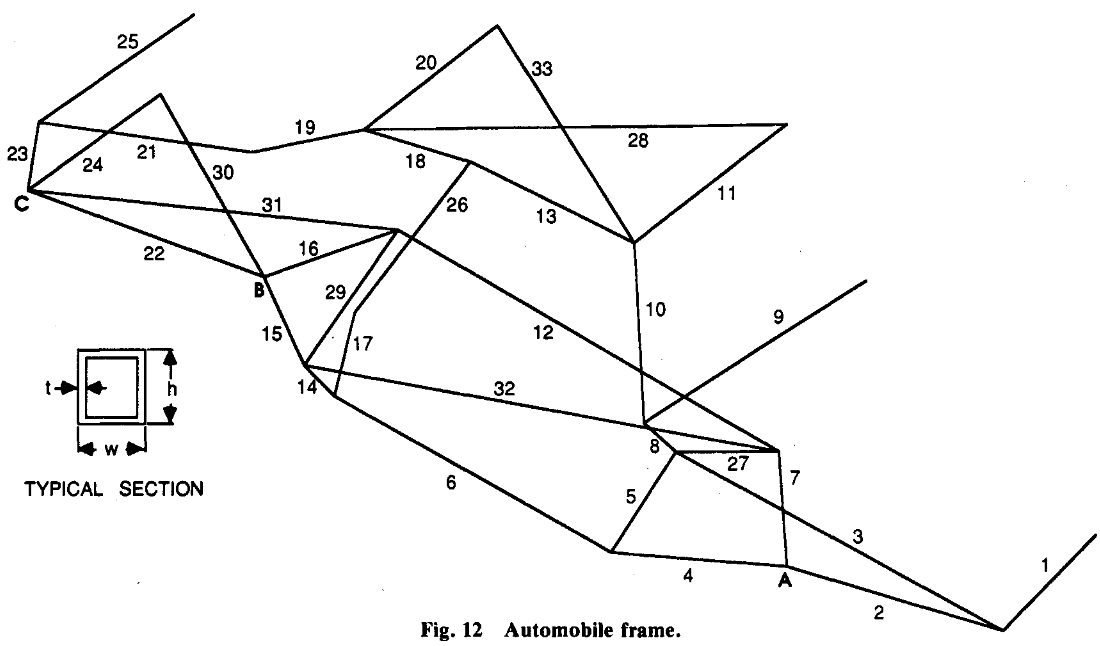

Fig. 12 Automobile frame.

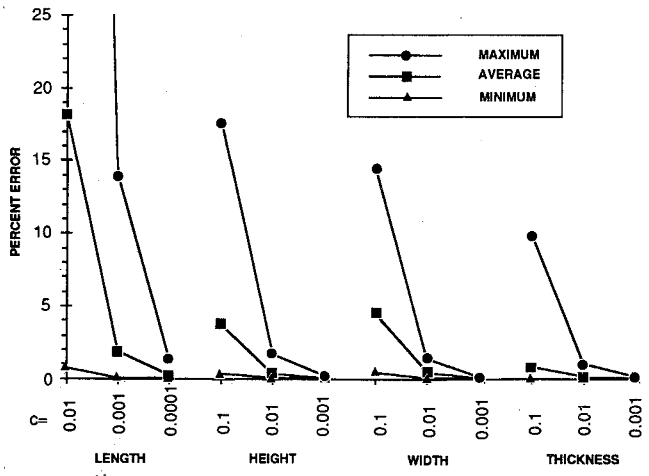

Fig. 13 Relative error as a function of finite difference parameter and design variable type.

Again, acceptable accuracy is attained when c is less than 0.01. Finally, for the case of the derivatives with respect to the element lengths, the accuracy is significantly worse than that obtained for the sizing variables for the same values of c. Relative errors greater than 60% are observed when $c = 0.01$. Again, however, acceptable accuracy is achieved when c is decreased to the smaller, but still reasonable, value of 0.0001.

As seen in Fig. 13, each of the design variables t, b, h, and ℓ exhibit different levels of accuracy. This can be attributed to the varying degrees of nonlinearity of the stiffness matrix with respect to these variables. For a thin walled box beam, the section properties are nearly linear in t and, therefore, the element level stiffness matrix is nearly linear in t. As a result, the accuracy of the displacement derivatives with respect to t is much better than that for b, h, and ℓ, for a given value of c. In general, the careful selection of design variables or other intermediate variables (e.g., beam section properties) for the derivative calculations will yield derivatives containing less error for any given value of c.

Concluding Remarks

The inaccuracy of the semianalytic method for computing static displacement derivatives with respect to shape and sizing design variables has been shown to be the result of errors in the pseudoload vectors. Two type of errors were identified. In the first case, the inaccuracy of the finite-difference operation results in a scaling of the pseudoload vector which, in turn, causes the derivatives to be uniformly scaled relative to their true values. In this situation, the errors in the derivatives depend only on the choice of the finite-difference parameter. In the second case, errors in the finite-difference operation lead to a distortion of the pseudoload vectors and nonuniform errors in the displacement derivatives. These errors may depend on the location (within the structure) of the element(s) associated with the design variable and the discretization of the structure, as well as on the finite-difference parameter c.

The results of the general error analysis were demonstrated both analytically and numerically for the case of an initially uniform cantilevered beam. The error in the tip displacement derivative with respect to the element height is independent of the element number and the number of elements used to model the beam. However, the accuracy of the tip displacement derivative with respect to element length varies with both the element number and the number of elements. In addition, we show that the cantilevered beam may display nonuniform relative errors for both shape and sizing variables that depend on the nature of the displacement fields resulting from the applied loads. These effects were also demonstrated for a typical automobile frame structure. For a given value of c, the derivative errors for shape design variables were generally larger than those for sizing variables.

We note that the cantilevered beam problem was specifically chosen to demonstrate and analyze the large errors that may occur when computing semianalytic derivatives. In contrast, the errors presented for the automotive frame, a standard design model used for structural optimization, are clearly errors that can be expected for realistic structures. For the problems studied here, the errors were adequately controlled through the choice of the finite-difference parameter. Although values as large as 0.01 were safely used for sizing variables, shape variables required values as small as 0.00001 to compute sufficiently accurate displacement derivatives. Acceptable accuracy was obtained in the automotive frame problem for both sizing and shape variables for $c = 0.0001$.

In practice, it is not clear how to handle the errors reported here for all problems. In some cases, it may be possible to choose intermediate design or property variables that appear linearly (or nearly linearly) in the stiffness matrix. This will greatly improve the accuracy of the finite-difference approximation. Generally, careful selection of the finite-difference parameter used for the computation of the stiffness matrix derivatives is a must. Large values of the finite-difference parameter may lead to large errors of the type analyzed here, whereas small values may lead to roundoff errors. To minimize roundoff effects, the finite-difference approximations should always be computed in double precision, as was done here. Guidelines for choosing c may be found in Ref. 9.

In some cases, we may not be able to choose a value of c such that the forward difference approximations have the desired accuracy. In these situations, another method must be used. Iterative correction schemes have been suggested[7]; unfortunately, they are not robust. The central difference formula, however, is clearly an effective way to control the finite-difference errors. In some cases, this may be done with only a slight increase in computational cost. Additional study may show central difference gradients to be the most practical approach for minimizing the errors in semianalytic derivatives.

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